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Certain methods for linear problems with inexact initial data ☆

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Abstract

Linear problems with inexact initial data are examined. The stopping rules for certain iterative methods designed for solving linear equations and a linear elimination problem are proposed and analysed. In particular, these methods are applicable to ill-conditioned and ill-posed problems. Numerical results are presented that demonstrate the efficiency of these methods.

MSC: 65F10; 65N22

Keywords: Linear problem; Iterative method; Ill-conditioned problem; Ill-posed problem; Inexact initial data

1. Introduction

In this paper two problems are considered: the problem of solving a linear equation and the elimination problem, which consists in the evaluation of a prescribed linear functional of a solution to a linear equation without calculating the solution itself. In particular, these problems may be ill-conditioned or ill-posed. It is supposed that some small errors (the perturbations) have been introduced into the initial data. Iterative methods for solving such problems are proposed and examined that use a priori information about these errors. The methods under consideration have been supplemented by the stopping criteria, in these criteria the information about the initial data errors have been used. It is proved that the approximate solutions calculated by these methods converge to the exact solutions of the original (without perturbation) problems

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when the errors go to zero. The numerical experiments confirmed the efficiency of these methods; ill-conditioned and ill-posed problems are the most advantageous case for the methods proposed.

2. Linear equation

The problem of solving linear equations with inexact initial data was analysed in a number of studies (in particular, see [5,7,8]). This section explores the possibility of applying Craig's method [6] to such equations, in particular, in the case of ill-conditioned and ill-posed problems.

Suppose that N and M are complex Hilbert spaces (the case when M and (or) N are finite-dimensional is also included); $A : M \to N$ is a linear bounded operator, and b is a given element of N. Consider the equation

$$Ax = b. \tag{2.1}$$

If Eq. (2.1) has a solution (possibly nonunique), then we denote by x_* its least-norm solution. It is well known that if Eq. (2.1) is solvable, such solution exists and is unique. It was shown in [1] that assuming the solvability of Eq. (2.1), Craig's method yields a sequence of elements of M that converges to x_*.

Consider the equation

$$\tilde{A}\tilde{x} = \tilde{b}, \tag{2.2}$$

where $\tilde{A}$ and $\tilde{b}$ are obtained from A and b by introducing small errors. Clearly, under the only assumption that Eq. (2.1) is solvable, even the existence of a solution to Eq. (2.2) cannot be guaranteed, and if such solution exists, one cannot guarantee its closeness to x_*.

It was indicated in [8] that, when Eq. (2.2) is solved by iterative methods in order to obtain an approximate solution to Eq. (2.1), it is reasonable to adjust the number of actual iteration steps to the accuracy of the initial data. Based on this idea, a number of iterative methods for approximate solution of Eq. (2.2) were proposed and examined (in particular, see [7]). This section pursues the same goal. We show that, if Eq. (2.1) is solvable and the stopping rule in Craig's method is chosen in a proper manner, the resulting approximate solution to Eq. (2.2) tends to x_* as $|A - \tilde{A}| \to 0$ and $|b - \tilde{b}| \to 0$. We propose two versions of the stopping rule.

Let us present the formulas for Craig's method as applied to Eq. (2.2) (without assuming its solvability). These formulas are relevant modifications of those given in [6].

The process begins with the following values:

$$\begin{aligned}
&x_1 = 0, \quad r_1 = \tilde{b}, \quad c_1 = r_1, \quad g_1 = \tilde{A}^* c_1, \quad \alpha_1 = (r_1, r_1)/(g_1, g_1); \quad \text{next, for } k = 2, 3, \ldots, \\
&x_k = x_{k-1} + \alpha_{k-1} g_{k-1}, \ \ r_k = \tilde{b} - \tilde{A} x_k, \ \ \beta_k = (r_k, r_k)/(r_{k-1}, r_{k-1}), \\
&c_k = r_k + \beta_k c_{k-1}, \quad g_k = \tilde{A}^* c_k, \quad \alpha_k = (r_k, r_k)/(g_k, g_k).
\end{aligned} \tag{2.3}$$

When $r_k = 0$ or $g_k = 0$, the process terminates. Here, $r_k = 0$ indicates that x_k is the desired solution to Eq. (2.2), and $r_k \neq 0$ coupled with $g_k = 0$ indicates that the equation has no solution.

Suppose that

$$|A - \tilde{A}||x_*| + |b - \tilde{b}| \leqslant \delta.$$

Theorem 1. *If*

$$2\delta|c_n| \leqslant |r_n|^2, \tag{2.4}$$

then $|x_* - x_{n+1}| \leqslant |x_* - x_n|$.

Theorem 2. *Suppose that Eq.* (2.1) *is solvable. Let* $\delta > 0$, *and let* $\Omega(\delta)$ *denote the set of* $\tilde{A}$ *and* $\tilde{b}$ *such that* $\delta \geqslant |\tilde{A} - A||x_*| + |\tilde{b} - b|$. *Suppose that process* (2.3) *continues as long as* (2.4) *is true. Then this process halts in a finite number n of steps* (*of course*, *n depends on* δ *and the problem solved*) *and*

$$\lim_{\delta \to 0} \left(\sup_{\Omega(\delta)} |x_* - x_n| \right) = 0.$$

The stopping rule described above can be modified as follows. Fix an arbitrary number D greater than 2. Suppose that $|A - \tilde{A}| \leqslant Q$, $|b - \tilde{b}| \leqslant R$. Introduce

$$\delta_n = Q|x_n| + R$$

and use, as a criterion for continuing the process, the relation

$$D\delta_n|c_n| \leqslant |r_n|^2, \tag{2.5}$$

instead of (2.4).

Theorem 3. *Suppose that Eq.* (2.1) *is solvable. Fix* D, $D > 2$. *Let* $Q \geqslant 0$, $R \geqslant 0$, *and* $Q + R > 0$. *Denote by* $\Omega(Q, R)$ *the set of* $\tilde{A}$ *and* $\tilde{b}$ *such that* $|\tilde{A} - A| \leqslant Q$ *and* $|\tilde{b} - b| \leqslant R$. *Suppose that process continues as long as* (2.5) *is true. Then this process halts in a finite number n of steps and*

$$\lim_{Q+R \to 0} \left(\sup_{\Omega(Q,R)} |x_* - x_n| \right) = 0.$$

Formula (2.5) has an advantage over (2.4). Specifically, there is no need to derive a preliminary estimate of $|x_*|$. However, in contrast to (2.5), formula (2.4) guarantees that $|x_* - x_{n+1}| \leqslant |x_* - x_n|$ for all n until the process halts.

3. Linear elimination problem

Suppose that in addition to Eq. (2.1) some element $g \in M$ is given. The elimination problem is as follows.

Consider the linear equation (2.1). It is required to evaluate the number $\sigma = (x, g)$; the solution x itself is not needed.

It is helpful to consider the dual problem, which is as follows. Consider the equation

$$A^* y = g. \tag{3.1}$$

It is required to evaluate the number $\sigma' = (b, y)$. In what follows, we assume that Eqs. (2.1) and (3.1) are solvable and their solutions may not be unique. In the latter case

$$\sigma = (x, g) = (x, A^* y) = (Ax, y) = (b, y) = \sigma' \tag{3.2}$$

and the value of σ is independent of particular solutions x and y to Eqs. (2.1) and (3.1). Note that if M and N are finite-dimensional and the solution to (2.1) is nonunique, then the fact that σ is independent of the choice of a solution to Eq. (2.1) is equivalent to the solvability of Eq. (3.1).

Consider a pair of equations (2.2) and

$$\tilde{A}^* \tilde{y} = \tilde{g}. \tag{3.3}$$

Here, $\tilde{g}$ (just as $\tilde{A}$, $\tilde{b}$) is obtained by introducing a small error in g. If only the solvability of Eq. (3.1) is assumed, then, obviously, the solvability of Eq. (3.3) cannot be guaranteed. If Eqs. (2.2) and (3.3) are solvable, one cannot guarantee that the result obtained will be close to σ.

The purpose of this section is to propose a method that processes the initial data $\tilde{A}$, $\tilde{b}$, and $\tilde{g}$ in such a manner that the resulting $\tilde{\sigma}$ satisfies $\tilde{\sigma} \to \sigma$ as $|\tilde{A} - A| \to 0$, $|\tilde{b} - b| \to 0$, and $|\tilde{g} - g| \to 0$ provided that Eqs. (2.1) and (3.1) are solvable. This method is based on the method for solving problems (2.1), (3.1), and (3.2) proposed in [2]. It is supplemented by the stopping criterion that uses bounds for $|\tilde{A} - A|$, $|\tilde{b} - b|$, and $|\tilde{g} - g|$ and is a modification of such criterion defined in Section 2.

Let us discuss in more detail the solvability assumption about the auxiliary (3.1), which is important for the subsequent analysis. Under this assumption, the problem of evaluating σ is numerically more stable than the problem of solving Eq. (2.1). Indeed, if Eqs. (2.2) and (3.3) are also solvable, then we have

$$\tilde{\sigma} - \sigma = (x, \tilde{g} - g) + (\tilde{b} - b, y) - ((\tilde{A} - A)x, y) + \theta, \tag{3.4}$$

where

$$\theta = (\tilde{x} - x, \tilde{g} - g - (\tilde{A}^* - A^*)y) = (\tilde{b} - b - (\tilde{A} - A)x, \tilde{y} - y). \tag{3.5}$$

If problems (2.1) and (3.1) are ill-conditioned or ill-posed, then the differences $\tilde{x} - x$ and $\tilde{y} - y$ may not be small. The first three summands on the right-hand side of (3.4) are small, and the fourth summand can be written (see (3.5)) as the scalar product of $\tilde{x} - x$ (or $\tilde{y} - y$) and a small vector. It follows that this summand in (3.4) is considerably smaller than $|\tilde{x} - x|$ and $|\tilde{y} - y|$.

Below, we list the required formulas of the method given in [2] as applied to Eqs. (2.2) and (3.3). The solvability of these equations is not assumed.

For $k = 1, 2, \ldots$, one calculates vector sequences b_k, g_k, p_k, and q_k and scalar sequences α_k, β_k, and γ_k by using the formulas

$$\begin{aligned}
&\beta_1 = |\tilde{A}\tilde{g}|, \quad \gamma_1 = |\tilde{A}^*\tilde{b}|, \quad b_1 = \tilde{A}\tilde{g}/\beta_1, \quad g_1 = \tilde{A}^*\tilde{b}/\gamma_1,\\
&\alpha_1 = (\tilde{A}g_1, b_1), \quad \hat{b}_1 = \tilde{A}g_1 - \alpha_1 b_1, \quad \hat{g}_1 = \tilde{A}^* b_1 - \overline{\alpha}_1 g_1,\\
&p_1 = \tilde{g}/\beta_1, \quad q_1 = \tilde{b}/\gamma_1, \quad \hat{p}_1 = g_1 - \alpha_1 p_1, \quad \hat{q}_1 = b_1 - \overline{\alpha}_1 q_1; \quad \text{next, for } k = 2, 3, \ldots,\\
&\beta_k = |\hat{b}_{k-1}|, \quad \gamma_k = |\hat{g}_{k-1}|, \quad b_k = \hat{b}_{k-1}/\beta_k, \quad g_k = \hat{g}_{k-1}/\gamma_k,\\
&\alpha_k = (\tilde{A}g_k, b_k), \quad \hat{b}_k = \tilde{A}g_k - \alpha_k b_k - \gamma_k b_{k-1},\\
&\hat{g}_k = \tilde{A}^* b_k - \overline{\alpha}_k g_k - \beta_k g_{k-1}, \quad p_k = \hat{p}_{k-1}/\beta_k, \quad q_k = \hat{q}_{k-1}/\gamma_k,\\
&\hat{p}_k = g_k - \alpha_k p_k - \gamma_k p_{k-1}, \quad \hat{q}_k = b_k - \overline{\alpha}_k q_k - \beta_k q_{k-1}.
\end{aligned} \tag{3.6}$$

Here, $\overline{\alpha}_k$ is the complex conjugate of α_k. (We will not go into simplification of these formulas.)

For simplicity, we assume that all β_k and γ_k are nonzero. Note that the vector sets $b_1, b_2, \ldots$ and $g_1, g_2, \ldots$ are orthonormal. For any k, it holds that $b_k = \tilde{A} p_k$ and $g_k = \tilde{A}^* q_k$.

Using the vectors $g_1, g_2, \ldots$ and $b_1, b_2, \ldots$, we construct approximations x_k and y_k of x and y, respectively, for $k = 1, 2, \ldots$:

$$x_k = \sum_{i=1}^{k} \xi_i g_i, \quad y_k = \sum_{i=1}^{k} \eta_i b_i.$$

Here,

$$\xi_i = (\tilde{b}, q_i), \quad \eta_i = (\tilde{g}, p_i).$$

The corresponding approximation σ_k of σ is given by

$$\sigma_k = (x_k, \tilde{g}) + (\tilde{b}, y_k) - (\tilde{A} x_k, y_k). \tag{3.7}$$

Denote by y_* the minimum-norm solution to Eq. (3.1). As shown in [2], when $\tilde{A} = A$, $\tilde{b} = b$, and $\tilde{g} = g$ (i.e., when one deals with exact Eqs. (2.1) and (3.1)), it holds that

$$\lim_{k \to \infty} x_k = x_*, \quad \lim_{k \to \infty} y_k = y_*.$$

Hence,

$$\lim_{k \to \infty} \sigma_k = \sigma.$$

Moreover, in this case $|\sigma_k - \sigma| = \mathrm{O}(|x_k - x_*| \times |y_k - y_*|)$. This underlines the appropriateness of formula (3.7) for calculating σ_k.

Note the following feature of formulas (3.6): they allow one to calculate the values x_k and y_k in parallel. Furthermore, the number of arithmetic operations required for calculating both vectors by the method under analysis is roughly the same as that required for calculating any of these vectors by a conjugate direction method. (The bulk of the work in calculating x_k (respectively, y_k) when x_{k-1} (respectively, y_{k-1}) has already been found is the multiplication of $\tilde{A}$ and $\tilde{A}^*$ by the corresponding vectors.)

Suppose that

$$|\tilde{A} - A||x_*| + |\tilde{b} - b| \leqslant \delta_x \quad \text{and} \quad |\tilde{A}^* - A^*||y_*| + |\tilde{g} - g| \leqslant \delta_y$$

(note that $|\tilde{A} - A| = |\tilde{A}^* - A^*|$).

Theorem 4. *If*

$$2\delta_x |q_n| \leqslant |(\tilde{b}, q_n)|, \tag{3.8}$$

then $|x_* - x_{n+1}| \leqslant |x_* - x_n|$. *If*

$$2\delta_y |p_n| \leqslant |(\tilde{g}, p_n)|, \tag{3.9}$$

then $|y_* - y_{n+1}| \leqslant |y_* - y_n|$.

Theorem 5. *Let Eqs.* (2.1) *and* (3.1) *be solvable. Suppose that* $\delta_x \geqslant 0$, $\delta_y \geqslant 0$, *and* $\delta_x + \delta_y > 0$. *Denote by* $\Omega(\delta_x, \delta_y)$ *the set of* $\tilde{A}$, $\tilde{b}$, *and* $\tilde{g}$ *such that* $\delta_x \geqslant |\tilde{A} - A||x_*| + |\tilde{b} - b|$ *and* $\delta_y \geqslant |\tilde{A} - A||y_*| + |\tilde{g} - g|$. *Assume that the process continues as long as conditions* (3.8) *and* (3.9) *are fulfilled. Then this process terminates in a finite number n of steps and*

$$\lim_{\delta_x+\delta_y\to 0}\left(\sup_{\Omega(\delta_x,\delta_y)} |\sigma - \sigma_n|\right) = 0.$$

Let us consider another stopping criterion. Fix arbitrary numbers D_x and D_y greater than 2. Assume that $|\tilde{A} - A| \leqslant Q$, $|\tilde{b} - b| \leqslant R_b$, and $|\tilde{g} - g| \leqslant R_g$. Define

$$\delta_{x,n} = Q|x_n| + R_b, \quad \delta_{y,n} = Q|y_n| + R_g.$$

Instead of (3.8) and (3.9), we will use, as a criterion for continuing the process, the following two relations:

$$D_x\delta_{x,n}|q_n| \leqslant |(\tilde{b}, q_n)|, \quad D_y\delta_{y,n}|p_n| \leqslant |(\tilde{g}, p_n)|. \tag{3.10}$$

For this criterion the theorem analogous to Theorem 5 is to be held.

Theorem 6. *Let Eqs.* (2.1) *and* (3.1) *be solvable. Fix numbers* D_x *and* D_y, *where* $D_x > 2$ *and* $D_y > 2$. *Suppose that* $Q \geqslant 0$, $R_b \geqslant 0$, $R_g \geqslant 0$, *and* $Q + R_b + R_g > 0$. *Denote by* $\Omega(Q, R_b, R_g)$ *the set of* $\tilde{A}$, $\tilde{b}$, *and* $\tilde{g}$ *such that* $|\tilde{A} - A| \leqslant Q$, $|\tilde{b} - b| \leqslant R_b$, *and* $|\tilde{g} - g| \leqslant R_g$. *Assume that the process continues as long as both conditions in* (3.10) *are fulfilled. Then this process terminates in a finite number n of steps and*

$$\lim_{Q+R_b+R_g\to 0}\left(\sup_{\Omega(Q,R_b,R_g)} |\sigma - \sigma_n|\right) = 0.$$

When the conditions in (3.10) are violated at significantly different n (due to, say, big differences in the perturbations R_b and R_g of the right-hand sides of Eqs. (2.1) and (3.1)), criterion (3.10) can be modified as described below. In this modification, the process continues as long as at least one of the relations in (3.10) is fulfilled. Furthermore, one fixes the approximations x_{n_x} and y_{n_y} obtained immediately before these relations have been violated, i.e., the approximations corresponding to the relation $D_x\delta_{x,n}|q_n| \leqslant |(\tilde{b}, q_n)|$ first violated at the iteration step $n_x + 1$ and to the relation $D_y\delta_{y,n}|p_n| \leqslant |(\tilde{g}, p_n)|$ first violated at the iteration step $n_y + 1$. Then, one calculates

$$\tilde{\sigma} = (x_{n_x}, \tilde{g}) + (\tilde{b}, y_{n_y}) - (\tilde{A}x_{n_x}, y_{n_y}).$$

4. Numerical experiments

Below, we present numerical results based on the theoretical analysis carried out in Sections 2 and 3. As a matrix of system (2.1) we used the Hilbert matrix

$$A = \|a_{ij}\| \in \mathbb{R}^{n\times m}, \quad a_{ij} = 1/(i + j - 1), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m,$$

which is known to be difficult for numerical treatment. The formulas were used under the assumption that $A : \mathbb{R}^m \to \mathbb{R}^n$ and the scalar products are conventional. The exact solutions to systems (2.1) and (3.1) were specified by the vectors $x=\|x_j\|$ and $y=\|y_i\|$, where $x_j=\sqrt{n_1}/(j+n_1-1)$, $y_i=\sqrt{m_1}/(i+m_1-1)$ for all i and j; n_1 and m_1 are fixed, $1\leqslant n_1\leqslant n$ and $1\leqslant m_1\leqslant m$. Since these solutions are linear combinations of the rows (columns) in A, they are minimum-norm solutions x_* and y_*, whatever the size $m\times n$ of A is. The factors $\sqrt{n_1}$ and $\sqrt{m_1}$ were taken to make norms of x_* and y_* near 1.

We proceeded from (2.1) and (3.1) to (2.2) and (3.3) by introducing errors into the entries of A, b, and g to some number of significant digits. Specifically, we set

$$\tilde{a}_{ij}=a_{ij}[1+\varepsilon_A\sin(2i^2+3j^2)],\quad \tilde{b}_i=b_i[1+\varepsilon_b\sin(5i^2)],\quad \tilde{g}_j=g_j[1+\varepsilon_g\sin(3j^2)],$$

where ε_A, ε_b, and ε_g are specified; thus the relative errors in $\tilde{a}_{ij}$, $\tilde{b}_i$, and $\tilde{g}_j$ do not exceed ε_A, ε_b, and ε_g, respectively. (These formulas imitate random errors.)

4.1. Numerical experiments for Section 2

Denote by ε_x the norm of the difference between the exact and approximate solution, by k_* the number of iterations determined by the stopping rule (2.4), and by $k_{\min}$ the number of iterations at which a minimum value of ε_x is reached. Set $n=5000$, $m=3000$, and $n_1=5000$; and let $\varepsilon_A=\varepsilon_b=\varepsilon$. Tables 1–4 present results obtained for various ε. Computations were also conducted with the use of the stopping rule (2.5). The results were found to almost coincide for the examples presented in Tables 1–4.

The examples presented suggest the following:

- The behaviour of the iterative process is such that the error first decreases down to some value (of order $\sqrt{\varepsilon}$ in the examples) and then increases.
- The stopping rule yields an error close to its minimum value (ε_x for $k=k_*$ is close to ε_x for $k=k_{\min}$).

Table 1

($\varepsilon=10^{-3}$)	
k	ε_x
$k_*=6$	0.228×10^{-1}
$k_{\min}=7$	0.213×10^{-1}
$k=15$	0.544×10^{1}
$k=100$	0.222×10^{4}

Table 2

($\varepsilon=10^{-4}$)	
k	ε_x
$k_*=7$	0.946×10^{-2}
$k_{\min}=10$	0.451×10^{-2}
$k=15$	0.999
$k=100$	0.370×10^{3}

Table 3

($\varepsilon = 10^{-6}$)	
k	ε_x
$k_* = 16$	0.526×10^{-3}
$k_{\min} = 16$	0.526×10^{-3}
$k = 35$	0.320
$k = 100$	0.108×10^{2}

Table 4

($\varepsilon = 10^{-10}$)	
k	ε_x
$k_* = 63$	0.8346×10^{-5}
$k_{\min} = 74$	0.8346×10^{-5}
$k = 95$	0.1158×10^{-4}
$k = 200$	0.1940×10^{-3}

Table 5

ε	10^{-3}	10^{-4}	10^{-6}
k_*	6	7	8
ε_σ	0.9996×10^{-3}	0.9941×10^{-4}	0.2443×10^{-5}
ε_x	0.275×10^{-1}	0.117×10^{-1}	0.450×10^{-2}
ε_y	0.219×10^{-1}	0.881×10^{-2}	0.350×10^{-2}

- When the error ε_x changes slowly near its minimum, the stopping rule gives a noticeable advantage in the number of iterations (see Table 4).
- The method is economical, because it requires a small number of steps even for large matrices.
- When the number of iterations is considerably greater than k_*, the results are poor even on qualitative level.

Some experiments for a Fredholm integral equation of the first kind also have been conducted, the results were analogous.

4.2. Numerical experiments for Section 3

Denote by ε_σ the relative error of the approximate value σ_k calculated at step k: $\varepsilon_\sigma = |1 - \sigma_k/\sigma|$. Here, we illustrate the numerical results obtained by using the stopping criteria (3.8), (3.9). Table 5 presents the number of iterations k_* and the quantities ε_σ, ε_x, and ε_y as a function of the perturbation ε in the initial data ($\varepsilon_g = \varepsilon$ also).

To summarize, we note the following:

- In all the variants of calculations, the behaviour of the process proposed is similar to that in Section 2.
- The calculated value of σ_k is considerably more accurate than the values of x_k and y_k; namely, its error for $k = k_*$ is of order ε. This corresponds to the property of the elimination problem indicated in Section 3: it is more stable with respect to perturbations in the initial data than the problem of solving a linear equation.

5. Conclusion

The numerical results presented show that the stopping rules provide good accuracy, while continued iteration leads to qualitatively incorrect result (although the residuals in the equation become small). This illustrates the efficiency of the stopping rules and used methods supplemented with these rules.

When the methods presented above are used in practice, all the formulas are implemented with rounding errors. It is interesting to analyse its influence.

The main results of this paper were published in [3,4].

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